Lautsch Finite Elemente GmbH

The
Shortest / Fastest / Finest
Introduction to the
Finite Element Method

The FE Method is simple when seen from an abstract point of view. You only have to do some projections. Like you have done in school, many years ago. Do you remember how to project a point onto a plane? The vector point - projection was perpendicular to the plane. This was done by a simple 2 by 2 linear system. 2 because it is a plane.

The projection point is exactly that location on the plane, which is nearest to the point you have started with. Now projection means best approximation and best approximation means projection.

The plane is an affine linear subspace of 3D space.

You really do the same to find the solution of the heat equation, only the items you work with are different.

We present 5 projections: 2 are known from your good school education,
3 from FE Analysis.

	projection = best approximation: point to line	point to plane	of a function	heat	linear static
What	point	point	function	PDE	PDE
source vs	3D space	3D space	function space	differentiable functions	vector of 3 functions
# dimensions of source vector space	3	3	inf	inf	inf
target vector space	line	plane	FE space	FE space	
# dimensions of target vector space	1	2	# nodes	# nodes	# nodes *3
inner product Euclidean:=	$< A , B > = A_1 * B_1 + A_2 * B_2 + A_3 * B_3$		$\int A * B$	$\int \text{grad } A * \text{grad } B$	$\int A\, D*S*D\, B$
distance	--------------	$\Vert A - B \Vert = \text{sqrt}\,(< A - B , A - B >)$		------------------	
linear system	----------$< a_1 A_1 + \ldots + a_n A_n , A_j > = < . , A_j >$			The Ritz-Galerkin method transforms the PDE to the linear system -----------	

Projection and best approximation

Let u be a Point (i.e. Element) of a Vector space named V .
Let u_i be a basis i = 1,...n of a finite dimensional subspace V_n and
let < , > be an inner product.

We define the distance $|| a – b || = \mathrm{sqrt}(< a-b , a-b >)$

$P = \sum a_i u_i$ is a point of V_n , P is the projection of u, if for all v of V_n

$< P-u, v > = 0.0$ (perpendicular conditions)

P is the Projection <=> P is the best approximation.

Proof „=>“ $|| P-u + v ||^2 = || P-u ||^2 + 2 < P-u, v > + ||v||^2 > || P-u ||^2$

Proof „<=“ $|| P-u ||^2 < || P-u + ev||^2 = || P-u || + 2e < P-u, v > + e^2||v||^2$

$0.0 < 2e < P-u, v > + e^2 ||v||^2$ for any e only if $< P-u, v > = 0.0$

Why good elements?

If all elements are good and small enough, it is guaranteed that your result is close to the exact solution.

In engineering practice you often have a lot of good elements and some bad ones. You obtain good results because your finite dimensional vectorspace is well suited.

Good elements => ~~<=~~ good approximation

Proof „=>“ The best approximation converges to the exact solution. (Ciarlet‘s book)
Proof „~~<=~~“ It is sufficient to give examples, i.e. to show that there are good approximations which are done with bad elements.