Lautsch Finite Elemente GmbH

Multigrid meshing

29.1.2025

Let points belong to parts, but only one part per point.

Elements are defined by points   1D Intervals,
2D triangles, quads, polygons …
3D tetras, pyras, prisms, hexas

Elements should be convex.

A set of elements tiles the set of parts which have to be meshed conformally.

Examples: 1D: Time steps for ODEs
2D: Delaunay, Voronoi, triangle- or quad tiling
3D: Delaunay, Voronoi, Hexa tiling, Tetra tiling

You need an idea how to represent the boundaries of the parts.

You need an idea how to avoid too many points and elements.

You need an idea how to achieve high approximation power.

boundaries of the parts

MMT creates its own part surface. This allows surface simplification and boolean operations. The most simple idea is to take a fine tiling and to delete those elements whose nodes have points of different parts. MMT splits these tetras by edge trisection in the last refinement step

avoid too many points and elements by adaptivity

MMT nodes before the last step are an adaptively selected subset of a fine BCC grid.

high approximation power by adaptivity

Barycentric limits for the last step, to avoid 0.0-volume Elements.

MMT is the extension of the h and h/2 stepsize control for ODE.

The Body Centered Cubic grid is a set of points. It is popular in crystallography, it is simply defined as the cubic grid + a (0.5, 0.5, 0.5) -translated copy of the cubic grid. All points have the same meaning.

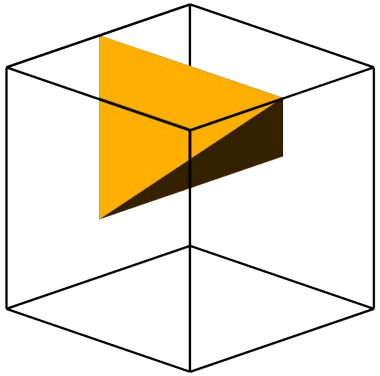
The elementary cell of this grid is celebrated at the Atomium in Brussels.

The space between the points may be filled by octahedrons. The edges are either of length 1.0 or 0.5 sqrt 3 .

Each octahedron may be split into two pyramids.

Or you can split each octahedron into 4 tetras. The octahedron has two diagonals of length sqrt 2 and one of length 1.0 . In order to obtain good element quality (*) we prefer the shortest one.

This Tetra is named cubic.



By edge midpoint splits this tetra can be split to 8 similar tetras.

By repeated splits, we obtain a mesh, with nodes arranged as in the BCC crystal grid. And we can perform local mesh refinement with limited loss of mesh quality.

To obtain a Finite Element mesh we have to introduce the boundaries of the material properties. Many authors suggest to BCC approximate the boundaries and to switch to clean elements in the last step.

(*)   1.0 normalized, Volume / (max edge length)**3

A BCC based mesh of cubic tetras is a Delaunay mesh.

The MMT nodes are a subset of the BCC-grid of cell-size   H / 2**q at the q th refinement step. H is the size of the longest edge of the initial mesh.

The MMT refinement terminates when no edges > RESOLUTION are marked for bisection or trisection.

THE MULTIGRID MARCHING TETRA METHOD

… terminates for simple parts.

BCC refinement: A single cubic Tetra covers all parts, its 4 points belong to the fluid which is treated like a part.

Loop: local refinement: Refine BCC tetras when MT refinement would fail.

Keep element quality > 0.155 by adding edge bisections.

All nodes are BCC.

Final local refinement step:

Each node is assigned to exactly one part.

create **visible** points: points on the part surface belong to the part and to the fluid.

Each tetra is assigned to exactly one part.

MT refinement: Tetras which are hit by the part geometry are split appropriately.

Keep element quality > user requirement by barycentric limits or other means.

Wikipedia: Bei komplexen Geometrien erreichen Mehrgitterverfahren schnell ihre Grenzen

Wittum
Lake Constance

Zienkiewicz
Error criterion

Trottenberg: Multigrid is too difficult

The **Multigrid Method** is optimal for the solution of Linear Systems for elliptic Problems

REVOLUTION of Numerical egineering: automatic meshing, using Delaunay‘s Method is broadly accepted

Various BCC methods create hierarchies of meshes

2024

2001

2002

2019

1994

2006

Founding LFE

Multigrid Marching Tetra

3D

2D

1990

1988

1986

1987 - 2006

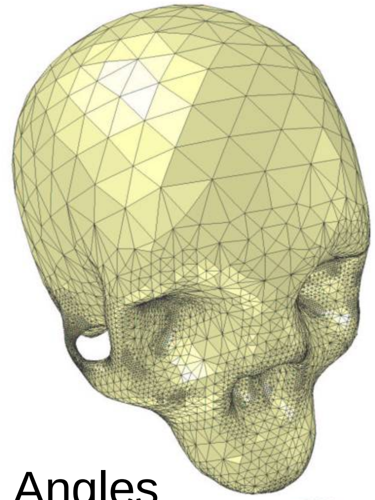
1984

Computer aided engineering at Daimler Benz: Waiting for Multigrid Methods

1982

My first numerical software

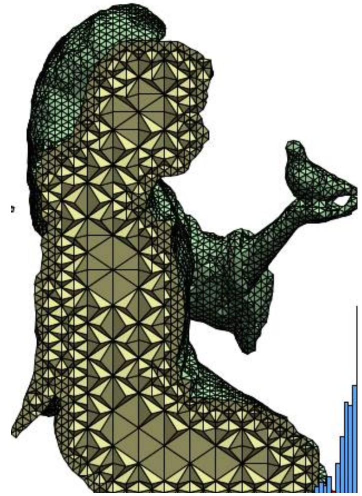
Learning
FE + Multigrid Method
at Bonn university, Prof. Alt



2002 Tetrahedral Mesh Generation for Deformable Bodies

Neil Molino, Robert Bridson, Ronald Fedkiw

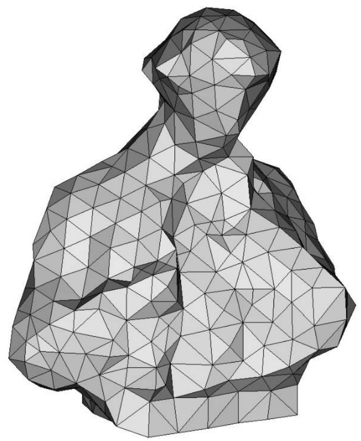
2007 Isosurface Stuffing: Fast Tetrahedral Meshes with Good Dihedral Angles



François Labelle, Jonathan Richard Shewchuk

2024 Constructing Tetrahedral Meshes No Matter How Ugly The CAD

Matt Staten, David Noble, Riley Wilson, Sandia National Laboratories

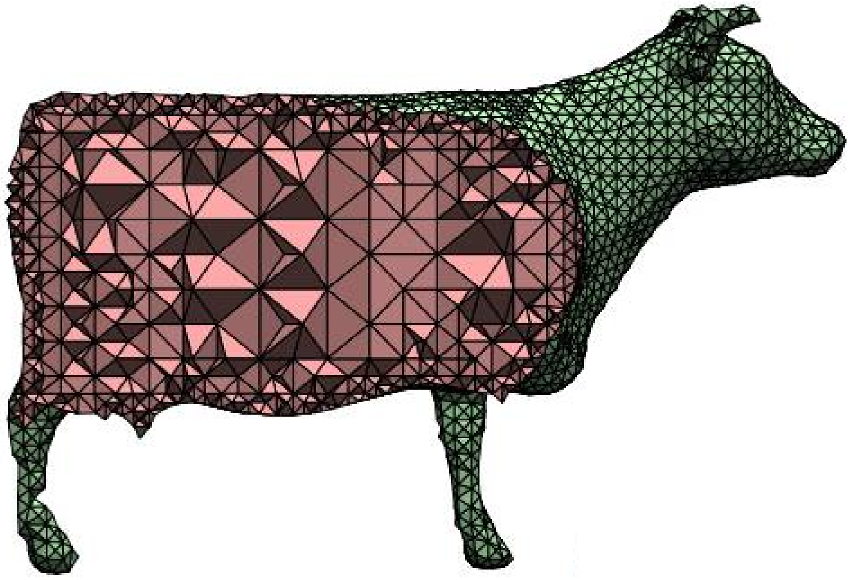


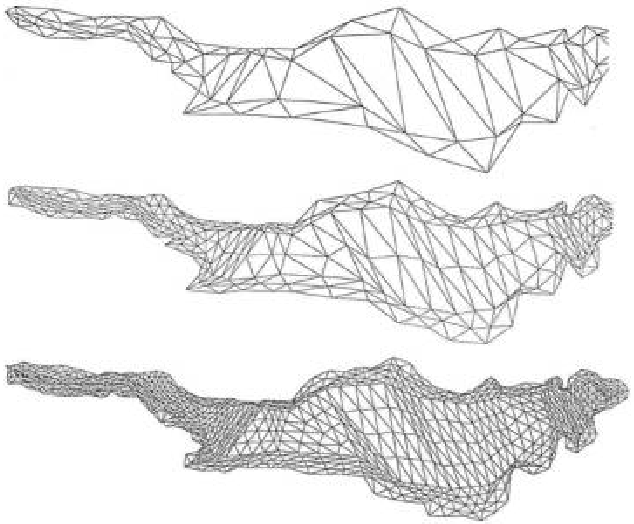
… and many similar methods:

Are they MULTIGRID in the sense that the linear system can be solved optimally?

If the answer is **YES**

GEOMETRIC MULTIGRID WORKS FOR ANY GEOMETRY





Automated 3D multigrid sequence of meshes -   2D multigrid sequence of meshes, created manually

2007 Labelle, Shewchuk   1990 Sautter, Wittum

The polyhedral twin

Prisms and pyramids are created in intermediate stages of the BCC and of the MT refinement. If we do not transform them to tetras and if we consider 4 cubic tetras with a common edge as an octahedron we obtain a

Conformal polyhedron mesh

Octahedron, hexahedron (= brick), prism, pyramid, tetra.

Each element is bounded by triangles and plane quads.

Element quality is bounded away from 0.0 .

Cubic tetra

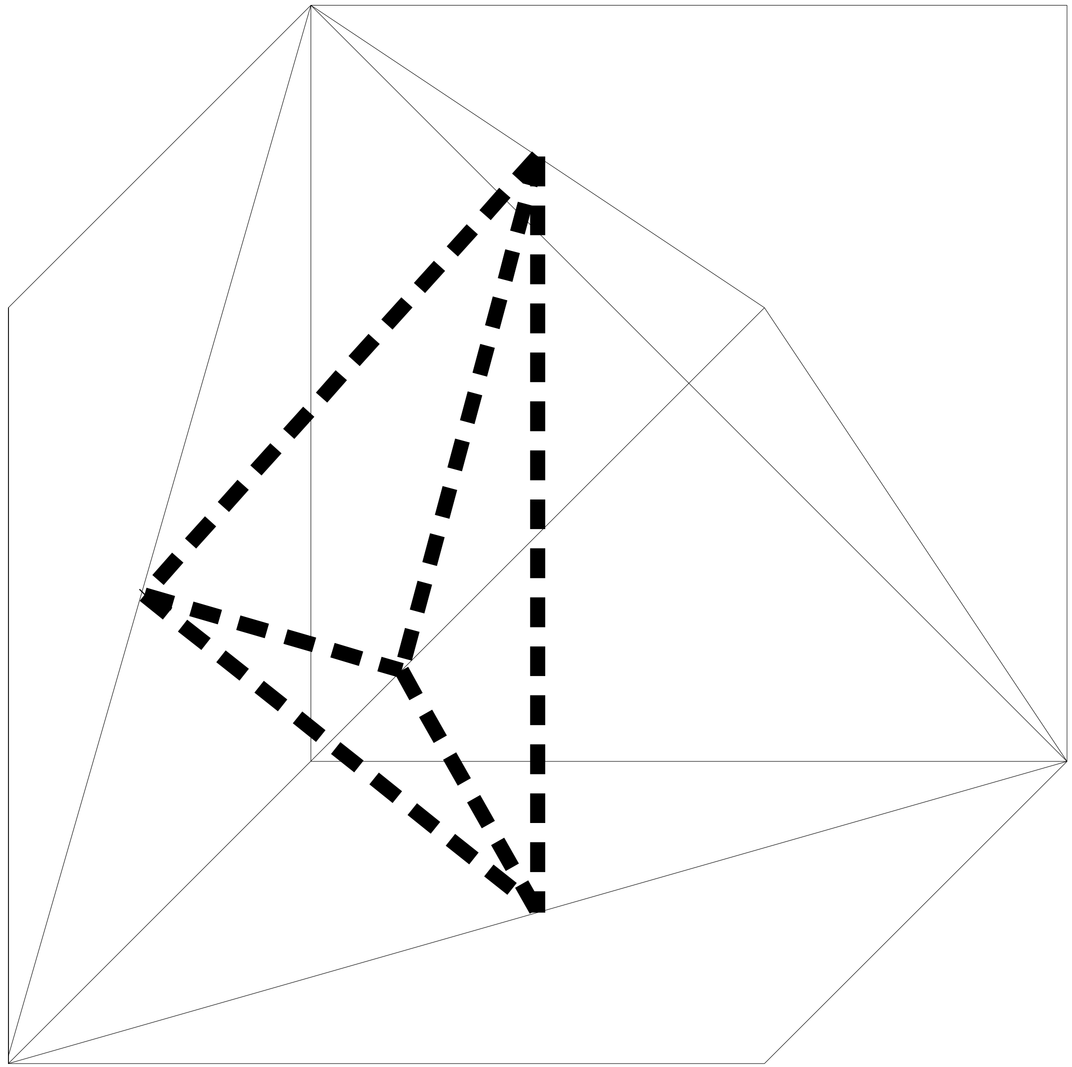
We never make use of the fact that any of the tetras is cubic. The 1 to 8 refinement just has to cut the internal octahedron to 4 tetras along the shortest diagonal.

Starting with a different tetra only results in worse tetra quality. Even starting with Platon‘s ideal tetra P.

$$P \longrightarrow 4\,P + 4\,B$$

$$B \longrightarrow 6\,B + 2\,P$$

Tetras	worst TET	av quality
1	1.00000	1.000
8	0.35355	0.677
64	0.35355	0.596
512	0.35355	0.576
4096	0.35355	0.571
32768	0.35355	0.569
262144	0.35355	0.569



This is B inside P inside the unique cube. The quality of B is 0.353   .
One edge of B is 1.0 and 5 edges are 0.5 sqrt 2   .